Ramification in Division Fields and Sporadic Points on Modular Curves

Hanson Smith

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Table of contents

Summary

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Let E be an elliptic curve over a number field K with good supersingular reduction at some prime $\mathfrak{p}$ living above the rational prime p.

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Call this valuation[1] μ. If $\mu \geq \frac{p}{p+1}$, then all the x-coordinates of p^n-torsion points have the same valuation, which is

$$\frac{-2}{p^{2n} - p^{2n-2}} = -2 \cdot \frac{1}{p^{2(n-1)}(p^2-1)}.$$

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Sporadic Points: Prime Power Level

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Sporadic Points: Composite Level

Let $N > 12$ be a positive integer not divisible by 6 and write $N = \prod_{i=1}^{k} p_i^{e_i}$ for the prime factorization. Suppose $E/\mathbb{Q}$ has good supersingular reduction at each p_i, then $j(E)$ does not correspond to a sporadic point on $X_1(N)$.

Moral

Being supersingular at primes dividing N can be an obstruction to having an N-torsion point defined over a number field of particularly low degree.

Context

Division/Torsion Fields

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- We can also ask about the possible torsion structures for elliptic curves over a number field with a given Galois group or degree.

Some Previous Work

Arithmetic of Torsion Fields: [Duke and Tóth, 2002]; [Adelmann, 2001]; [Kraus, 1999], [Cali and Kraus, 2002], [Freitas and Kraus, 2018]; [González-Jiménez and Lozano-Robledo, 2016].

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Uniform Boundedness +: [Merel, 1996]; Oesterlé's proof: [Derickx et al., 2017, Appendix A]; [Parent, 1999]; [Lozano-Robledo, 2018].

Valuations of Points

Canonical Subgroups

$P = (x, y) \rightsquigarrow -\frac{x}{y} \in \hat{E}$

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It could have one side (one-slope case) corresponding to all p-torsion elements in $\hat{E}$ having the same valuation, or it could have two sides (two-slope case), corresponding to a subgroup of $\hat{E}[p]$ of order p having larger valuation.

The Two-Slope Case

Notice μ is the valuation of the coefficient corresponding to sums of products of $p^2 - p$ roots.

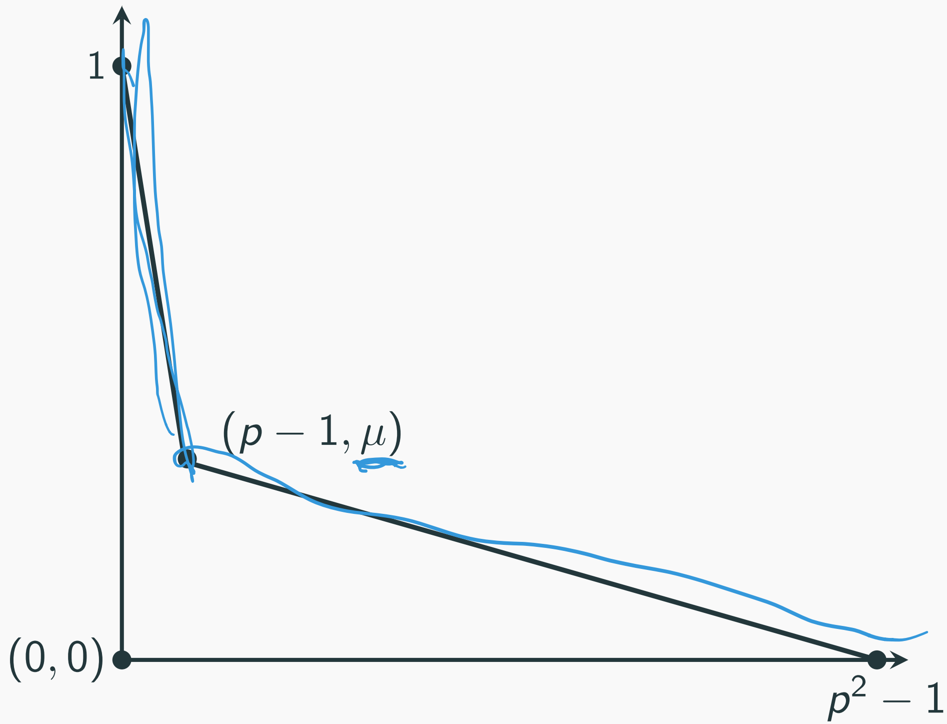

Figure 1: The Newton polygon for the polynomial $\prod_{\hat{P} \in \hat{E}[=p]} (T - \hat{P})$

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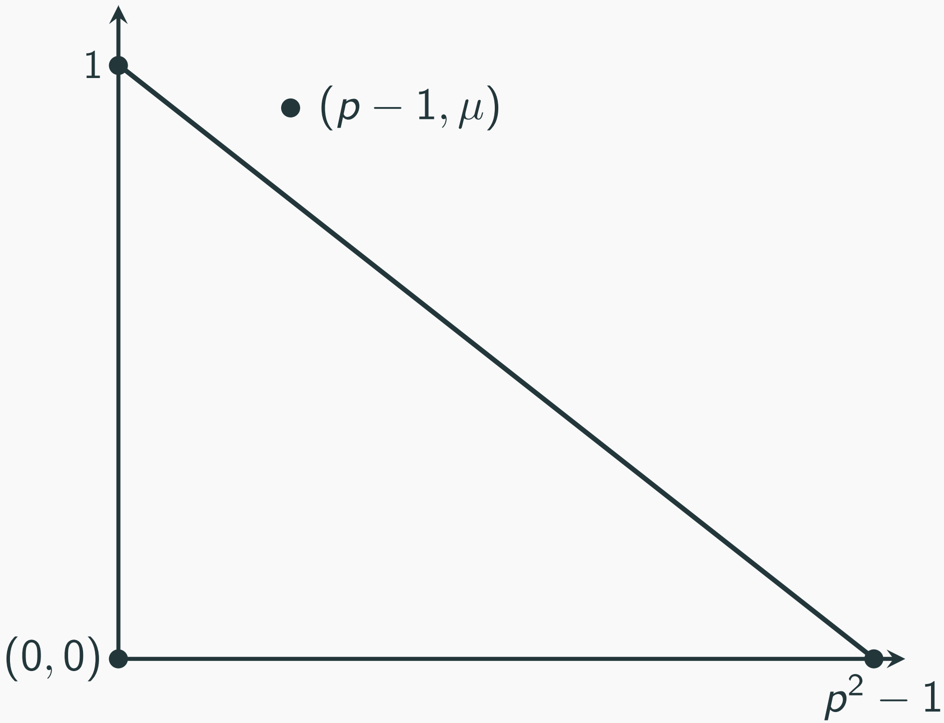

Figure 2: The Newton polygon for the polynomial $\prod_{\hat{P} \in \hat{E}[=p]} (T - \hat{P})$

Canonical Subgroups

In the ‘two-slope case’ people say that E has a *canonical subgroup* at $\mathfrak{p}$. This is because the subgroup of $\hat{E}$ with larger valuation is a canonical lift of the kernel of Frobenius. This subgroup is very important to those who study overconvergent modular forms and well-studied in that context.

Valuations and Canonical Subgroups

Recall, μ is the valuation of the coefficient of $x^{\frac{p^2-p}{2}}$ in the p^{th} division polynomial. Equivalently, it is the valuation of the coefficient of T^p in $[p]T$. When there is a canonical subgroup, then you can think of μ as the sum of the valuations of elements of $\hat{E}[=p]$ that are **not** in the canonical subgroup.

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If $\mu < \frac{p}{p+1}$, then there is a canonical subgroup. The elements that are not in it have valuation $\frac{\mu}{p^2-p}$ and the elements that are in it have valuation $\frac{1-\mu}{p-1}$.

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First off, if there is no canonical subgroup, we "just divide by p^2." So a 125-torsion element has valuation $\frac{1}{5^4(5^2-1)}$.

$\vdots$

25-tors $\qquad \frac{1}{5^2(5^2-1)}$

$\big\downarrow [5]$

5-tors $\qquad \frac{1}{5^2-1}$

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Even when there is a canonical subgroup, for points that are not above it we still divide by p^2. So a 125-torsion element $\hat{P}$ such that $[5^2]\hat{P}$ is not in the canonical subgroup has valuation $\frac{\mu}{5^4(5^2-5)}$.

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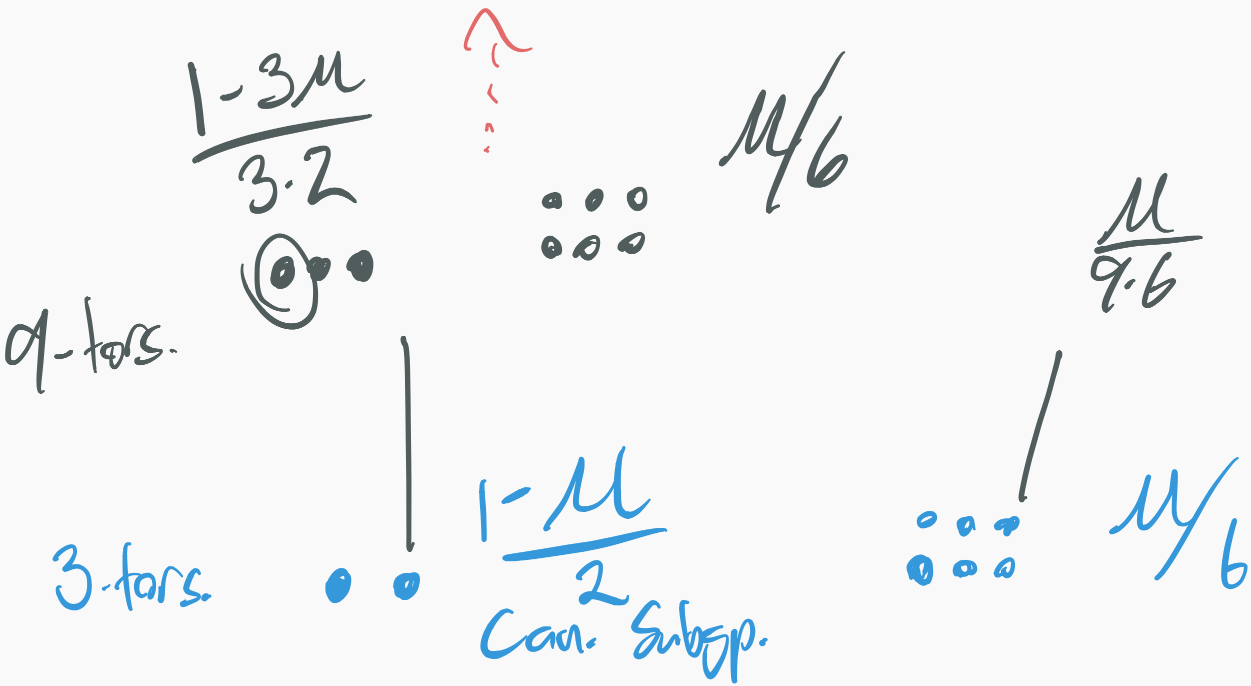
Higher Order Subset of Larger Valuation

Interestingly, we have a phenomenon that is similar to the canonical subgroup in some ways occurring for higher power torsion. When μ is small enough, then in the fibers over (p^{th} roots of) elements in the canonical subgroup, $[p]^{-1}\hat{Q}$, there is a subset of p-elements with larger valuation.



Partial Theorem Statement

For $n > 1$, let $s \in \mathbb{Z}^{\geq 0}$ be the smallest integer such that $\mu \geq \frac{1}{p^s(p+1)}$. If $n \leq s+1$, then either

$$v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{1 - p^{n-1}\mu}{p^{n-1}(p-1)} \quad \text{or} \quad v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{\mu}{p^{2m}(p^2-p)}, \tag{1}$$

where m is the smallest non-negative integer such that $v_{\mathfrak{p}}\left([p^m]\,\hat{P}\right) = \frac{\mu}{p^2-p}$.

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$$v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{1 - p^{s}\mu}{p^{2n-s-2}(p-1)} \quad \text{or} \quad v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{\mu}{p^{2m}(p^2-p)}, \tag{2}$$

where m is as above.

Proof Ideas

Main Idea

Stare at the power series for the multiplication-by-p map in the formal group of E at $\mathfrak{p}$ for a long time. Because E is supersingular at $\mathfrak{p}$ this is equivalent to staring at the p^{th} division polynomial.

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Let $\pi_{\mathfrak{p}}$ be a uniformizer at $\mathfrak{p}$. The multiplication-by-p map has the form

$$[p]T = pf(T) + \pi_{\mathfrak{p}}^{\mu} g\left(T^{p}\right) + h\left(T^{p^2}\right),$$

where f, g, and h are power series without constant coefficients and with $f'(0), g'(0), h'(0)$ all units.

After a little work we see that we must compare $pv(\hat{P}) + \mu$ and $p^2 v(\hat{P})$ where $\hat{P}$ is the image of a point of $E[= p^n]$ in the formal group. We also have that the minimum of these two values is greater than or equal to $\hat{Q}$, where $Q \in E[= p^{n-1}]$.

Application to Sporadic Points on $X_1(N)$

Degrees of Minimal p^n-Torsion Fields

The work above gives the minimal ramification necessary to have a p^n-torsion point in terms of the valuation of a coefficient of the p^{th} division polynomial. This yields a lower bound on the degree of a field over which a p^n-torsion point is defined.

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So to preclude sporadic points on $X_1(p^n)$, compare the lower bound with have with an upper bound on the $\mathbb{Q}$-gonality of the modular curve $X_1(p^n)$. There is also some dotting of i's and crossing of t's with additive reduction resolving to good supersingular reduction and Weber functions.

Prime Power Level Sporadic Points

Let E be an elliptic curve that is supersingular at some prime above p with no canonical subgroup ($\mu \geq \frac{p}{p+1}$), then $j(E)$ does not correspond to a sporadic point on $X_1(p^n)$ for any $n > 0$.

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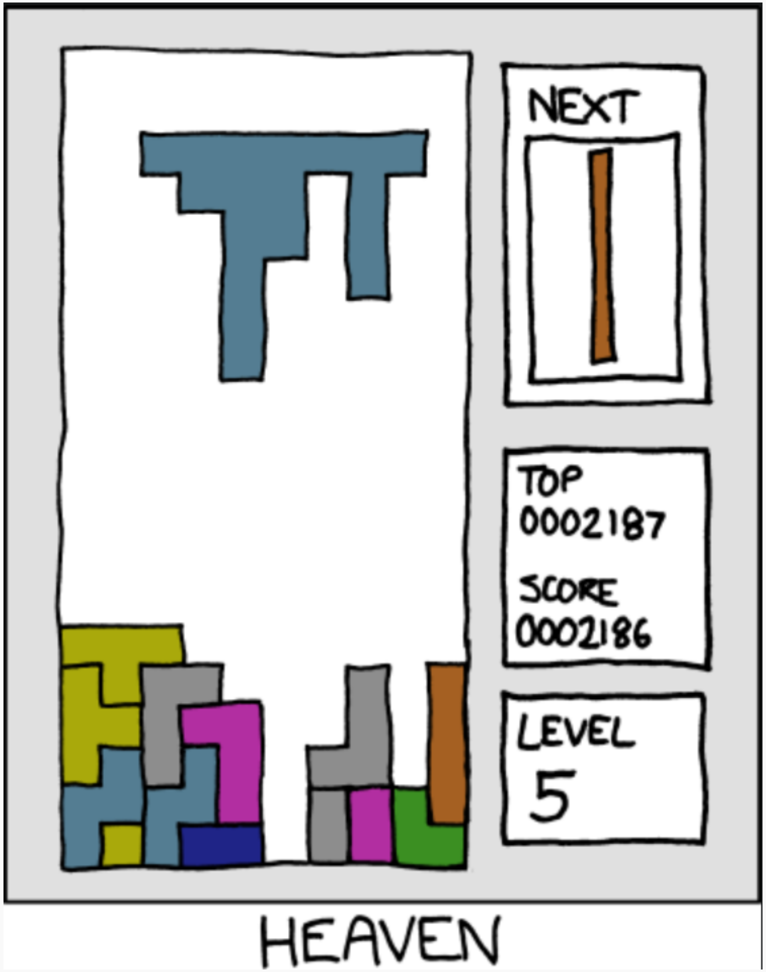
Elliptic curves with a canonical subgroup are "less supersingular" because, like ordinary elliptic curves, they have a canonical lift of the kernel of Frobenius. Hence, if one was willing to speak imprecisely (which I always am), we could say that the most supersingular elliptic curves do not correspond to sporadic points.

https://xkcd.com/888/



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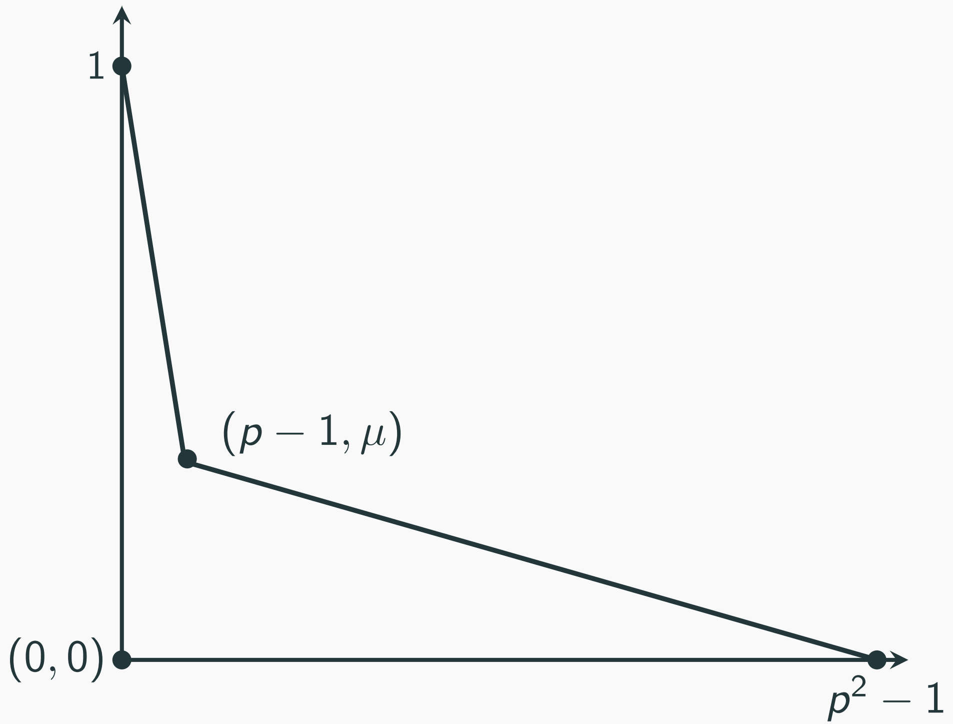

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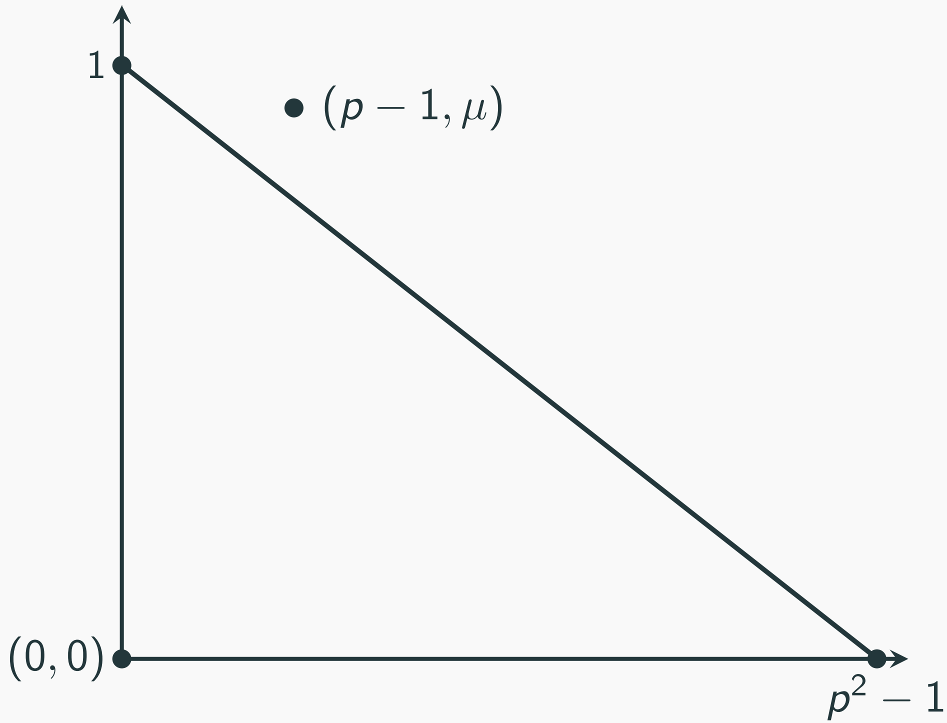

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Even when there is a canonical subgroup, for points that are not above it we still divide by p^2. So a 125-torsion element $\hat{P}$ such that $[5^2]\hat{P}$ is not in the canonical subgroup has valuation $\frac{\mu}{5^4(5^2-5)}$.

Higher Order Subset of Larger Valuation

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Interestingly, we have a phenomenon that is similar to the canonical subgroup in some ways occurring for higher power torsion. When μ is small enough, then in the fibers over (p^{th} roots of) elements in the canonical subgroup, $[p]^{-1}\hat{Q}$, there is a subset of p-elements with larger valuation.

Partial Theorem Statement

For $n > 1$, let $s \in \mathbb{Z}^{\geq 0}$ be the smallest integer such that $\mu \geq \frac{1}{p^s(p+1)}$. If $n \leq s+1$, then either

$$v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{1 - p^{n-1}\mu}{p^{n-1}(p-1)} \quad \text{or} \quad v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{\mu}{p^{2m}(p^2-p)}, \tag{1}$$

where m is the smallest non-negative integer such that $v_{\mathfrak{p}}\left([p^m]\,\hat{P}\right) = \frac{\mu}{p^2-p}$.

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$$v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{1 - p^{s}\mu}{p^{2n-s-2}(p-1)} \quad \text{or} \quad v_{\mathfrak{p}}\left(\hat{P}\right) = \frac{\mu}{p^{2m}(p^2 - p)}, \tag{2}$$

where m is as above.

Proof Ideas

Main Idea

Stare at the power series for the multiplication-by-p map in the formal group of E at $\mathfrak{p}$ for a long time. Because E is supersingular at $\mathfrak{p}$ this is equivalent to staring at the p^{th} division polynomial.

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Let $\pi_{\mathfrak{p}}$ be a uniformizer at $\mathfrak{p}$. The multiplication-by-p map has the form

$$[p]T = pf(T) + \pi_{\mathfrak{p}}^{\mu} g\left(T^{p}\right) + h\left(T^{p^2}\right),$$

where f, g, and h are power series without constant coefficients and with $f'(0), g'(0), h'(0)$ all units.

After a little work we see that we must compare $pv(\hat{P}) + \mu$ and $p^2v(\hat{P})$ where $\hat{P}$ is the image of a point of $E[=p^n]$ in the formal group. We also have that the minimum of these two values is greater than or equal to $\hat{Q}$, where $Q \in E[=p^{n-1}]$.

Application to Sporadic Points on $X_1(N)$

Degrees of Minimal p^n-Torsion Fields

The work above gives the minimal ramification necessary to have a p^n-torsion point in terms of the valuation of a coefficient of the p^{th} division polynomial. This yields a lower bound on the degree of a field over which a p^n-torsion point is defined.

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So to preclude sporadic points on $X_1(p^n)$, compare the lower bound with have with an upper bound on the $\mathbb{Q}$-gonality of the modular curve $X_1(p^n)$.

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So to preclude sporadic points on $X_1(p^n)$, compare the lower bound with have with an upper bound on the $\mathbb{Q}$-gonality of the modular curve $X_1(p^n)$. There is also some dotting of i's and crossing of t's with additive reduction resolving to good supersingular reduction and Weber functions.

Prime Power Level Sporadic Points

Let E be an elliptic curve that is supersingular at some prime above p with no canonical subgroup ($\mu \geq \frac{p}{p+1}$), then $j(E)$ does not correspond to a sporadic point on $X_1(p^n)$ for any $n > 0$.

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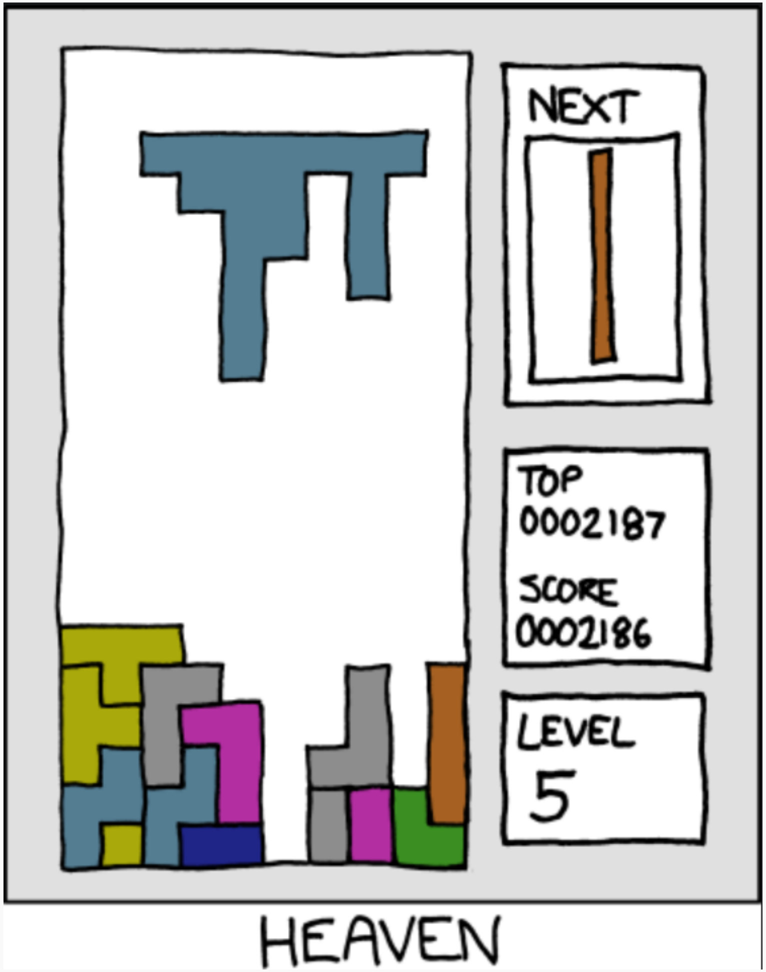
Elliptic curves with a canonical subgroup are “less supersingular” because, like ordinary elliptic curves, they have a canonical lift of the kernel of Frobenius. Hence, if one was willing to speak imprecisely (which I always am), we could say that the most supersingular elliptic curves do not correspond to sporadic points.

https://xkcd.com/888/



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